Massive gravity and Caustics: A definitive covariant constraint analysis

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Motivation and Goals

- de Rham-Gabadadze-Tolley massive gravity: no Boulware-Deser ghost
- Presence of superluminalities (cf Galileons in the decoupling limit): intense debate in the community
- First fully non-linear propagation analysis
  - Method of characteristic surfaces
  - Reliance on covariant constraint analysis
- Potential propagation pathologies...
Covariant constraint analysis

Spacelike characteristic surfaces

Discussion
Covariant constraint analysis

Spacelike characteristic surfaces

Discussion
GENERAL SETTING

- First order Cartan formalism in 4D
  - 4 vierbein 1-forms $e^m := e_\mu^m dx^\mu$ (16 fields)
  - 6 connection 1-forms $\omega^{mn} := \omega_\mu^{mn} dx^\mu$ (24 fields)

- dRGT action (4 fiducial vierbein 1-forms $f^m := f_\mu^m dx^\mu$)

$$S = -\frac{1}{4} \int \epsilon_{mnrs} e^m e^n \left[ d\omega^{rs} + \omega^r_t \omega^{ts} \right]$$

$$- m^2 \int \epsilon_{mnrs} e^m \left[ \frac{\beta_0}{4} e^n e^r e^s + \frac{\beta_1}{3} e^n e^r f^s + \frac{\beta_2}{2} e^n f^r f^s + \beta_3 f^n f^r f^s \right]$$
EQUATIONS OF MOTION

- Zero torsion condition

\[ T^m := \nabla e^m := de^m + \omega^m{}_n e^n \approx 0 \]

- Einstein equations

\[ G_m := G_m - m^2 t_m \approx 0 \]

- Einstein 3-forms

\[ G_m := \frac{1}{2} \epsilon_{m n r s} e^n [d \omega^{r s} + \omega^r{}_t \omega^{t s}] \]

- Mass term 3-forms

\[ t_m := \epsilon_{m n r s} \left[ \beta_0 e^n e^r e^s + \beta_1 e^n e^r f^s + \beta_2 e^n f^r f^s + \beta_3 f^n f^r f^s \right] \]

- 40 eoms for 40 dynamical fields
**Primary Constraints**

- Space-time decomposition of a $p$-form ($p < 4$)

  \[ \theta := \dot{\theta} + \theta \]

  where $\dot{\theta} \wedge dt = 0$

- Primary constraints: purely spatial part of the eoms

  \[ T^m = de^m + \omega^m{}_n e^n \approx 0 \]

  \[ G_m \approx m^2 t_m \]

- $12 + 4 = 16$ constraints
SECONDARY CONSTRAINTS

- Symmetry constraint
  \[ G_{[m\epsilon_n]} = \frac{1}{2} \epsilon_{mnrsc} \nabla T^s \approx 0 \]
  So \( t_{[m\epsilon_n]} \approx 0 \) and generically
  \[ \mathcal{F} := e_m f^m \approx 0 \]

- Vector constraint
  \[ \nabla G_m = \frac{1}{2} \epsilon_{mnrsc} T^n \left[ d\omega^{rs} + \omega^r_t \omega^{ts} \right] \approx 0 \]
  So \( \nabla t_m \approx 0 \) which reduces to
  \[ \mathcal{V} := \epsilon_{mnrsc} M^{mn} K^{rs} \approx 0 \]
  where \( M^{mn}(e^r, f^s) \) and \( K^{mn} := \omega^{mn} - \bar{\omega}^{mn}(e^r) \)

- 6+4=10 constraints
TERTIARY CONSTRAINTS

- Curl of symmetry constraint
  \[ \nabla F := K_{mn} e^m f^n \approx 0 \]
  where the purely spatial part is not new!

- Curl of vector constraint
  \[ \nabla V := \epsilon_{mnr} M^{mn} \nabla K^{rs} + \cdots \approx 0 \]

  \nabla K^{rs}: no time derivatives on-shell

  Scalar constraint:
  \[ S := \nabla V \approx 0 \]

- 3+1=4 constraints
- 40-16-10-4=10 first order dofs OR 5 physical dofs
Covariant constraint analysis

Spacelike characteristic surfaces

Discussion
Spacelike characteristic surfaces

- Spacelike characteristic surfaces: hypersurfaces which cannot be used to as initial data surfaces for the eoms (cf A. Terana’s parallel session talk: Cauchy breakdown)
- Analogous problem for first order differential equations:

\[ a(y, t)\dot{y} + b(y, t) = 0 \]

with \( y(t_0) = y_0 \)

- IF \( a(y_0, t_0) = 0 \): impossible to evolve the differential equation for the given initial conditions at \( t_0 \)!
- For mGR PDEs: non-invertibility of the coefficient of the highest order derivatives on some spacelike hypersurface \( \Sigma \)
Other interpretations

- $\Sigma$ is the world-sheet of a superluminal shock wave-front

- Equivalently: propagation of a superluminal wave-front in the infinite frequency limit over some mean-field solution of the eoms

- Causal structure: Light-cone of the theory larger than the dynamical metric light-cone

- !Potentially dangerous (cf M. Trodden’s plenary talk)
Analysis

- Investigation of the eoms+constraints
- Suppose $\Sigma$ is spacelike characteristic surface (with timelike normal $\xi_\mu$)
- Propagation of a shock wave-front along which first order derivatives are discontinuous

\[
\partial_\mu e^m_\nu \big|_{\Sigma^+} - \partial_\mu e^m_\nu \big|_{\Sigma^-} := \xi_\mu \epsilon^m_\nu \\
\partial_\mu \omega^{mn}_\nu \big|_{\Sigma^+} - \partial_\mu \omega^{mn}_\nu \big|_{\Sigma^-} := \xi_\mu \omega^{mn}_\nu
\]

- Compute the discontinuities in the eoms+constraints
- See whether the $\epsilon^m_\nu$ and $\omega^{mn}_\nu$ are allowed to be non-zero
RESULTS

- Characteristic equation:
  \[ \chi \left( \begin{array}{c} e^m_{\nu} \\ w^{mn}_{\nu} \end{array} \right) \approx 0 \]

- Characteristic matrix \( \chi \) depends on the initial conditions (compatible with the constraints) given on \( \Sigma \) OR equivalently on the mean-field solution of the eoms over which shocks propagate

- Invertibility is not warranted (and even dubious generically)
RESULTS

\( l_o^m K_{mn} f^n \omega_{omn} \approx 0 \)

\( 2\epsilon_{mnrs} l_o^m (\beta_1 e^n + \beta_2 f^n) K^{rs} \omega_{oo} - \epsilon_{mnrs} M^{mn} \omega_{or} \approx 0 \)

\( \epsilon_{mnrs} (\beta_1 e^m e^t - 2\beta_2 e^{(m f^t)} - 3\beta_3 f^m f^t) (K^{nr} \omega_{or} - K^r_s \omega_{or}^n) \)

\( + 2\epsilon_{mnrs} (\beta_1 l_o^{[m} e^{t]} - \beta_2 l_o^{(m} f^{t)}) K^{nr} K^s_t \omega_{oo} \)

\( + 2m^2 \epsilon_{mnrs} l_o^m \left( 4\beta_0 \beta_1 e^n e^r e^s + 3(\beta_1^2 + 2\beta_0 \beta_2) e^n e^r f^s \right. \)

\( \left. + 6\beta_1 \beta_2 e^n f^r f^s + (\beta_1 \beta_3 + 2\beta_2^2) f^n f^r f^s \right) \omega_{oo} \)

\( - 3\epsilon_{mnrs} \beta_3 f^m f^n \left( \rho^{rs} + 2m^2 \xi^{[r} [\tau^s] - \frac{1}{2} \tau^t_t e^s ] \right) \omega_{oo} \)

\( - 4\beta_2 l_o^m \tilde{G}_m \omega_{oo} - 2\epsilon_{mnrs} \beta_1 l_o^m e^n \tilde{R}^{rs} \omega_{oo} \approx 0 \)
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Discussion
**How bad is this?**

- $\Sigma$ can be any spacelike hypersurface!

- Light-cone of the theory completely flat?

- Maybe non-invertibility occurs only in strong-coupling regimes where the theory cannot be trusted anyway...

- Maybe for some region of parameter space OR some fiducial background choice, $\chi$ is generically invertible...

- Thorough analysis difficult...
CONCLUSION

- First covariant constraint analysis of dRGT mGR
- Study of characteristic surfaces
- The theory has potentially pathological behavior unless...
  - Strong-coupling and quantum corrections save the day
  - Magic happens (in the characteristic matrix)